Heat Conduction Toolbox – HC Toolbox
X22B10T0
James V. Beck, Filippo de Monte, et al. – December 5, 2011

X22B10T0 = X22B(t1)0T0 problem

fX22B10T0, fvX22B10T0

Heat conduction functions for the X22B10T0 case.

Syntax (Matlab)

Td = fX22B10T0(xd, td)
Td = fX22B10T0(xd, td, n)
Td = fvX22B10T0(xd, td)
Td = fvX22B10T0(xd, td, n)

Description

fX22B10T0 (xd, td) provides the dimensionless temperature distribution Td at a given dimensionless location xd from the heated surface, between 0 and 1, and at a given dimensionless time td with a default accuracy of $10^{-6}$ for a 1D Cartesian finite slab subject to a time-independent surface heat flux at one side (step change) and thermally insulated at the opposite side.

fvX22B10T0 (xd, td) provides the dimensionless temperature distribution Td in a matrix form for the same problem when xd and td are vectors defining the dimensionless locations and times of interest, respectively. If xd and td are vectors, length(xd) = n and length(td) = m, where \([m,n] = \text{size}(Td)\). The default accuracy is of $10^{-6}$.

fX22B10T0 (xd, td, n) and fvX22B10T0 (xd, td, n) give the dimensionless temperature distribution Td for the same problem with an accuracy of $10^{-n}$ ($n = 1, 2, \ldots$).

$n = \text{integer} (1, 2, \ldots, 10, \ldots)$ for solution accuracy; $n = 15$ gives an accuracy of one part in $10^{15}$ (machine accuracy)
Examples

Example 1

\[ T_d = fX22B10T0(0, .1, 2) \]

\[ T_d = 0.356829372437085 \]

Example 2

\[ T_d = fX22B10T0(0, .1, 15) \]

\[ T_d = 0.356826246008654 \]

Example 3

\[ n = 15 \]

\[ n = 15 \]

\[ x_d = [0.1, 0.2, 0.3] \]

\[ x_d = [0.10000000000000, 0.20000000000000, 0.30000000000000] \]

\[ t_d = [0.1, 0.2, 0.3] \]

\[ t_d = [0.10000000000000, 0.20000000000000, 0.30000000000000] \]

\[ T_d = fVX22B10T0(x_d, t_d, n) \]

\[ T_d = [0.265710758078575, 0.411546515326888, 0.528355069707451,
0.191930076891576, 0.330554257779286, 0.44484545330602,
0.134245086447475, 0.261793430683321, 0.372166721740129] \]
Example 4

```matlab
>> n=15
n =
    15
>> xd=[0.1 0.5 0.7]'
xd =
    0.100000000000000
    0.500000000000000
    0.700000000000000
>> td=[0.01 0.2]'
td =
    0.010000000000000
    0.200000000000000
>> Td=fvX22B10T0(xd,td,n)
Td =
    0.039928245674849   0.411546515326888
    0.000014352414313   0.158352196668220
    0.000000019773817   0.094884894165447
```
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Schematic

Nomenclature

\( k \)  thermal conductivity \((W/(m \, ^\circ C))\)
\( L \)  slab thickness \((m)\)
\( q_0 \)  surface heat flux \((W/m^2)\)
\( t \)  time \((s)\)
\( T \) temperature \((^\circ C)\)
\( x \)  Cartesian space coordinate \((m)\)
\( \alpha \)  thermal diffusivity \((m^2/s)\)

Governing equations

\[
\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad (0 < x < L; \ t > 0)
\]

\[
-k \left( \frac{\partial T}{\partial x} \right)_{x=0} = q_0 \quad (t > 0)
\]

\[
\left( \frac{\partial T}{\partial x} \right)_{x=L} = 0 \quad (t > 0)
\]

\[
T(x,0) = 0 \quad (0 < x < L)
\]
Exact short-time solution ([1, p. 112, Eq. (4), for the X22B01T0 case], [2, p. 196, Eq. (6.52b) with $m = 0$ and Eq. (6.53c)])

$$T(x,t) = 2 \frac{q_0 \sqrt{\alpha t}}{k} \text{ierfc} \left( \frac{x}{2 \sqrt{\alpha t}} \right)$$

$$+ 2 q_0 L \left( \frac{\alpha t}{L^2} \right)^{1/2} \sum_{m=1}^{\infty} \text{ierfc} \left( \frac{2mL+x}{2\sqrt{\alpha t}} \right) + \text{ierfc} \left( \frac{2mL-x}{2\sqrt{\alpha t}} \right)$$

(0 ≤ $x$ ≤ $L$; $t$ ≥ 0)

where the first term on the RHS is the well-known 1D Cartesian semi-infinite solution of the X20B1T0 problem [1, p. 75, Eq. (6)]. In addition, ierfc($z$) is the complementary error function integral defined as [2, p. 498, Eq. (E.9a)]

$$\text{ierfc}(z) = \int_{z}^{\infty} \text{erfc}(t) dt$$

The relationship between the ierfc($z$) function and the complementary error function erfc($z$) returned by the Matlab function erfc is [2, p. 501, Eq. (E.14a)]

$$\text{ierfc}(z) = e^{-z^2}/\sqrt{\pi} - z \text{erfc}(z)$$

The short-time solution comes from the application of Laplace transform to the governing equations. It is valid at any time but it is computationally convenient at short times.

If the time $t$ at a given location $x$ is less than the 1D deviation time $t^{(\text{dev})}$ [3, p. 5935, Eq. (19)], that is,

$$t^{(\text{dev})} = \frac{0.1}{n\alpha} \left( 2L - x \right)^2$$

(n = 1, 2, ..., 15),

we can consider only the first term in the above exact short-time solution with errors less than $10^{-n}$. (Note that $n = 2$ is for visual comparison, while $n = 15$ is for verification purposes of large numerical codes.) Then, we have

$$T(x,t) \approx 2 \frac{q_0 \sqrt{\alpha t}}{k} \text{ierfc} \left( \frac{x}{2\sqrt{\alpha t}} \right)$$
This indicates that, at short times (less than the deviation time listed before), the thermal deviation effects due to the homogeneous ‘inactive’ boundary condition at \( x = L \) are negligible (less than \( 10^{-n} \)) and the 1D finite slab can be considered as 1D semi-infinite along \( x \) and subject to a time-independent surface heat flux.

**Exact large-time solution ([2, p. 205, Eqs. (6.87) and (6.95)], [4, p. 2559, Eq. (25a)])**

\[
T(x,t) = \frac{q_0 L}{k} \left[ \frac{\alpha t}{L^2} + \frac{1}{2} \left( \frac{x}{L} \right)^2 - \frac{x}{L} + \frac{1}{3} \right] - 2 \sum_{m=1}^{\infty} \frac{1}{(m\pi)^2} \cos \left( \frac{m\pi x}{L} \right) e^{-\left(\frac{m\pi}{L}\right)^2 \frac{\alpha t}{L^2}}
\]

\((0 \leq x \leq L; \ t \geq 0)\)

The large-time solution comes from the application of separation-of-variables (SOV) method to the governing equations. It is valid at any time but it is computationally convenient at large times.

If the time \( t \) is greater than a characteristic time \( t^{(c)} \), that is,

\[
t^{(c)} = \frac{n \ln 10 \ L^2}{\pi^2 \ \alpha} \quad \quad (n = 1, 2, \ldots, 15),
\]

we can consider at any location \( x \) only one term in the above exact large-time solution with errors less than \( 10^{-n} \). Then, we have

\[
T(x,t) \approx \frac{q_0 L}{k} \left[ \frac{\alpha t}{L^2} + \frac{1}{2} \left( \frac{x}{L} \right)^2 - \frac{x}{L} + \frac{1}{3} - \frac{2}{\pi^2} \cos \left( \frac{\pi x}{L} \right) e^{-\frac{\pi^2 \alpha t}{L^2}} \right]
\]
Dimensionless quantities
The \( \Pi \) theorem states that, with four basic dimensions, mass, \([M]\), length, \([L]\), time, \([t]\) and temperature, \([T]\), a reduction of up to four may be hoped for in the number of the variables (seven) involved in the X22B10T0 problem. Therefore, we have a total of three dimensionless groups

\[
T_d = \frac{T}{q_0 L / k}, \quad x_d = \frac{x}{L}, \quad t_d = \frac{\alpha t}{L^2} = F_0,
\]

where \( x_d \in [0,1] \) and \( F_0 \) is the well-known Fourier number.

Computation of the dimensionless temperature solution at any location and time

\[
T_d(x_d, t_d) \approx \begin{cases} 
2\sqrt{t_d} \text{erfc} \left( \frac{x_d}{2\sqrt{t_d}} \right) & \text{for } 0 \leq t_d < t_d^{(p)} \\
& \\
& t_d^{(p)} \leq t_d \leq t_d^{(c)} \\
& t_d > t_d^{(c)} 
\end{cases}
\]

where

- \( t_d^{(p)} \) is the dimensionless partitioning time. In this case, it is exactly the same as the 1D deviation time defined before. In dimensionless form, we have

\[
t_d^{(p)} = \frac{0.1}{n} (2 - x_d)^2 \quad (n = 1, 2, \ldots, 15)
\]

For \( n = 15 \), we have a machine accuracy, i.e. \( 10^{-15} \), but the user can choose whichever accuracy s/he likes \( (10^{-n}) \).

- \( M \) is the maximum number of terms in the summation given by [2, p. 153, Subsection 5.2.1]
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\[ M = \text{ceil} \left( \frac{n \ln 10}{\pi^2 t_d} \right)^{1/2} \quad (n = 1, 2, \ldots, 15) \]

where the function “ceil(A)” rounds the number A to the nearest integer greater than or equal to A. For \( n = 15 \), we have a machine accuracy but the user can choose the accuracy desired (10\(^{-n}\)). The tail \( S_M \) of the summation (2\(^{nd}\) expression of \( T_d(x_d, t_d) \)) is given by [2, p. 153, Eq. (5.13)]

\[ S_M = \frac{1}{M \pi^{3/2}} \text{ierfc} \left( M \pi t_d^{1/2} \right) \]

It is always less than 10\(^{-n}\) [5].

- \( t_d^{(c)} \) is the dimensionless characteristic time defined before. In dimensionless form, we have

\[ t_d^{(c)} = \frac{n \ln 10}{\pi^2} \quad (n = 1, 2, \ldots, 15) \]
Matlab function: fX22B10T0.m

% fX22B10T0.m function
% Revision History
% 11 12 11 written by James V. Beck and Filippo de Monte
% calling sequence:
% none called
function Td=fX22B10T0(xd,td,n)
if td==0
Td=0;
elseif 0<td<(0.1/n)*(2-xd)^2;
arg=xd/sqrt(4*td);
ierfc_arg=(1/sqrt(pi))*exp(-arg^2)-arg*erfc(arg);
else
M=round(sqrt(n*log(10)/(td*pi^2))); %maximum number of terms
Td=td+((1/3)-xd+xd^2/2); % Start X22B10T0 case
for m=1:M
beta=m*pi; % m-th eigenvalue
Xf=cos(beta*xd); % m-th eigenfunction
Td=Td-2*exp(-beta^2*td)*Xf/beta^2;
end % for m
end % if

Matlab function: \( \text{fvX22B10T0.m} \)

\[
\text{fvX22B10T0.m function}
\]
\[
\text{Revision History}
\]
\[
\text{11 16 11. Added option of xd and td being vectors by James V. Beck}
\]
\[
\text{11 12 11 written by James V. Beck and Filippo de Monte}
\]
\[
\text{calling sequence:}
\]
\[
\text{none called}
\]
\[
\text{function Td=fvX22B10T0(xd,td,n)}
\]
\[
\text{sizex=length(xd);}
\]
\[
\text{xdv=xd;}
\]
\[
\text{sizet=length(td);}
\]
\[
\text{tdv=td; \% end vector option}
\]
\[
\text{Td=zeros(sizex,sizet); \% Preallocating Arrays for speed}
\]
\[
\text{for it=1:sizet}
\]
\[
\text{td=tdv(it);}
\]
\[
\text{for ix=1:sizex}
\]
\[
\text{xd=xdv(ix);}
\]
\[
\text{if td==0}
\]
\[
\text{Td(ix,it)=0;}
\]
\[
\text{elseif 0<td<(0.1/n)*(2-xd)^2;}
\]
\[
\text{arg=xd/sqrt(4*td);}
\]
\[
\ierfc_arg=(1/sqrt(pi))*exp(-arg^2)-arg*erfc(arg); \% ierfc function}
\]
\[
\text{Td(ix,it)=2*sqrt(td)*ierfc_arg; \% X20B1T0}
\]
\[
\text{else}
\]
\[
\text{M=round(sqrt(n*log(10)/(td*pi^2)))); \% maximum number of terms}
\]
\[
\text{Td(ix,it)=td+((1/3)-xd+xd^2/2); \% Start}
\]
\[
\text{X22B10T0 case}
\]
\[
\text{for m=1:M}
\]
\[
\text{beta=m*pi; \% m-th eigenvalue}
\]
\[
\text{Xf=cos(beta*xd); \% m-th eigenfunction}
\]
\[
\text{Td(ix,it)=Td(ix,it)-2*exp(-beta^2*td)*Xf/beta^2;}
\]
\[
\text{end \% for m}
\]
\[
\text{end \% if}
\]
\[
\text{end \% ix}
\]
\[
\text{end \% it}
\]
Dimensionless temperature values for various dimensionless times and distances.

2D Plot
Dimensionless temperature values for various dimensionless times and distances.

3D Plot
### Dimensionless temperature values for various dimensionless times and distances.

*Table*

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References

5. de Monte, F., Beck, J. V., Tail for $S_M = \sum_{m=M}^{\infty} \frac{e^{-\beta_m^0 t}}{\beta_m^2}$ and $\sum_{m=M}^{\infty} \frac{e^{-\beta_m^0 t}}{\beta_m^4}$, $\beta_m = m\pi$
   Unpublished Notes, December 2011.